ON WAVE SOLUTION OF FIELD EQUATIONS IN EINSTEIN'S UNIFIED FIELD THEORY*

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ABSTRACT. A static non-symmetric tensor field $g_{\mu\nu}$ with 4 symmetric components S_{kk}, S_{mm}, S_{nn}, S_{ll} $(k, m, n, l = 1, 2, 3, 4)$ and one anti-symmetric component a_{mn} (the rest being all zeros), involving either the coordinate x_k or the coordinates x_k, x_m is made non-static by changing x_k into $x_k + \epsilon x_l$ where ϵ is a constant. On solving the relevant field equations in the 'strong' form it is found that such a wave solution is generally spurious, being transformable into corresponding static solution. It is, however, observed that if the field is not spherically symmetric non-trivial solutions in wave form may be constructed.

1. INTRODUCTION

In a recent paper (Ghosh. 1957) a type of non-static solution of Einstein's field equations in 'strong' form was studied with respect to the tensor field $g_{\mu\nu}$ having the following structure :

$$g_{\mu\nu} = \begin{bmatrix} S_{kk} & 0 & 0 & a_{kl} \\ 0 & S_{mm} & a_{mn} & 0 \\ 0 & -a_{mn} & S_{nn} & 0 \\ -a_{kl} & 0 & 0 & S_{ll} \end{bmatrix} \begin{pmatrix} k, m, n, l \\ = 1, 2, 3, 4 \end{pmatrix}, \qquad \ldots \quad (1.1)$$

where S_{kk}, S_{ll}, a_{kl} are functions of $(x_k + \epsilon x_l)$, ϵ being a constant and S_{mm}, S_{nn}, a_{mn} are each expressible as the product of a function of $x_k + \epsilon x_l$ and a function of x_m of the type

$$\begin{aligned} S_{mm} &= \phi_{mm}(x_k + \epsilon x_l)\psi_{mm}(x_m), \\ S_{nn} &= \phi_{mm}(x_k + \epsilon x_l)\psi_{nn}(x_m), \\ a_{mn} &= \phi_{mn}(x_k + \epsilon x_l)\sqrt{\psi_{mm}\psi_{nn}}. \end{aligned} \qquad \ldots \quad (1.2)$$

On completing the solution of the relevant field equations we were led to make the following observations :

(a) The ψ's in (1.2) satisfy an auxiliary differential equation involving a parameter λ which can take negative, positive or zero values. When λ is negative the equation gives $\psi_{mm} = 1$, and $\psi_{nn} = \sin^2 x_m$, which corresponds to a spherically symmetric field. When λ is positive we get $\psi_{mm} = 1$, $\psi_{nn} = \sinh^2 x_m$ and for $\lambda = 0$ the ψ's are transformable into constants.

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(b) A solution of ϕ's with $\lambda \neq 0$ in the above wave form is generally spurious as it is transformable into the corresponding solution for the static field.

(c) When $\lambda = 0$ new static solutions are obtained, but with further restrictions, non-trivial solutions in wave form can be constructed, where the field components are functions of the argument $x_k + \epsilon x_l$ only.

The object of the present paper is to consider a special case of the above in which the tensor field (1.1) has only a single non-zero anti-symmetric element a_{mn} (the other one being zero) and to show that the above statements are in general maintained.

2. SYSTEM OF FIELD EQUATIONS WITH $a_{kl} = 0$, $a_{mn} \neq 0$.

To obtain the field equations in terms of non-vanishing Γ's we make use of the general formulae given in one of my earlier papers (Ghosh, 1955). The relevant equations are $R_{kk} = 0$, $R_{ll} = 0$, $R_{\underline{kl}} = 0$, $R_{mm} = 0$, $R_{\underset{\vee}{mn}} = 0$ of which the first three are linearly connected. Without going into details of the computation which is similar to that contained in my previous paper (Ghosh, 1957) we at once start with the following simplified set of 4 mutually independent field equations :

$$-A_{k,k} + \tfrac{1}{2}A_k(\gamma_{kkk} + \gamma_{llk}) - \tfrac{1}{2}(A^2_k + \overline{B}^2_k) = 0, \qquad \dots \quad (2.1)$$

$$\begin{aligned}&\gamma_{llk,k} + \gamma_{llk}(A_k - \tfrac{1}{2}\gamma_{kkk} + \tfrac{1}{2}\gamma_{llk})\\ &+ \epsilon^2 \frac{S_{kk}}{S_{ll}} [\gamma_{kkk,k} + \gamma_{kkk}(A_k - \tfrac{1}{2}\gamma_{llk} + \tfrac{1}{2}\gamma_{kkk})] = 0,\end{aligned} \qquad \dots (2.2)$$

$$\begin{aligned}&B_{k,k} + B_k(A_k - \tfrac{1}{2}\gamma_{kkk} + \tfrac{1}{2}\gamma_{llk})\\ &+ \epsilon^2 \frac{S_{kk}}{S_{ll}} [\overline{B}_{k,k} + B_k(A_k - \tfrac{1}{2}\gamma_{llk} + \tfrac{1}{2}\gamma_{kkk})] = \frac{2S_{kk}\phi_{mn}f(x_m)}{(\phi^2_{mm}+\phi^2_{mn})\psi_{mm}},\end{aligned} \dots \quad (2.3)$$

$$\begin{aligned}&A_{k,k} + A_k(A_k - \tfrac{1}{2}\gamma_{kkk} + \tfrac{1}{2}\gamma_{llk})\\ &+ \epsilon^2 \frac{S_{kk}}{S_{ll}} [A_{k,k} + A_k(A_k - \tfrac{1}{2}\gamma_{llk} + \tfrac{1}{2}\gamma_{kkk})] = \frac{-2S_{kk}\phi_{mm}f(x_m)}{(\phi^2_{mm}+\phi^2_{mn})\psi_{mm}}\end{aligned} \quad \dots \quad (2.4)$$

In the above

$$\begin{aligned}A_k &= (\phi_{mm}\phi_{mm,k} + \phi_{mn}\phi_{mn,k})/(\phi^2_{mm} + \phi^2_{mn}),\\ B_k &= (\phi_{mn}\phi_{mm,k} - \phi_{mm}\phi_{mn,k})/(\phi^2_{mm} + \phi^2_{mn}).\\ f(x_m) &= \tfrac{1}{2}[\gamma_{nnm,m} - \tfrac{1}{2}\gamma_{nnm}(\gamma_{mmm} - \gamma_{nnm})],\end{aligned} \qquad \dots \quad (2.5)$$

the symbols like γ_{kkk}, γ_{nnm} denoting respectively $S_{kk,k}/S_{kk}$ and $S_{nn,m}/S_{nn}$. Further, we have the condition

$$(\phi^2_{mm} + \phi^2_{mn})(\phi^2_{mm} - \phi^2_{mn}) \neq 0, \qquad \dots \quad (2.6)$$

which ensures uniqueness in the solution of Γ's.

The system of equation $\Gamma^{\nu}_{\underset{\vee}{\mu\nu}} = 0$ are identically satisfied and does not contribute any additional equation.

In the equations (2.3) and (2.4), there is a part $f(x_m)/\psi_{mm}$ which is a function of x_m only, while the rest involves only $x_k + \epsilon x_l$ and therefore, for consistency, we must have the auxiliary equation

$$4f(x_m) = \lambda \,.\, \psi_{mm}(\lambda, \text{ constant}) \tag{2.7}$$

This equation has been discussed in an earlier paper (Ghosh, 1956), leading to the statement (a).

3. SOLUTION OF FIELD EQUATIONS WHEN $\lambda \neq 0$

To solve equations (2.1—4) we adopt the procedure similar to that followed in my previous paper (Ghosh, 1957).

Introduce two new variables P, Q, functions of $x_k + \epsilon x_l$, defined by

$$S_{kk} = \frac{A^2{}_k(\phi^2{}_{mm}+\phi^2{}_{mn})^{\frac{1}{2}}}{P}, \quad S_{ll} = \frac{\epsilon^2 A^2{}_k(\phi^2{}_{mm}+\phi^2{}_{mn})^{\frac{1}{2}}}{Q} \tag{3.1}$$

then

$$\epsilon^2 S_{kk}/S_{ll} = Q/P \tag{3.2}$$

and

$$\gamma_{kkk} = 2A_{k,k}/A_k + A_k - P_k/P,$$

$$\gamma_{llk} = 2A_{k,k}/A_k + A_k - Q_k/Q. \tag{3.3}$$

Substituting from (3.3) in (2.1) we get

$$A_{k,k} - \frac{1}{2} A_k \left(\frac{P_k}{P} + \frac{Q_k}{Q}\right) + \frac{1}{2} (A_k{}^2 - \bar{B}_k{}^2) = 0 \,. \tag{3.4}$$

Using (3.4) let us rewrite (3.3) as

$$\gamma_{kkk} = \frac{Q_k}{Q} + \frac{B^2{}_k}{A_k}, \tag{3.5}$$

$$\gamma_{llk} = \frac{P_k}{P} + \frac{\bar{B}^2{}_k}{A_k}.$$

If we insert (3.2) and (3.5) in (2.2), it takes the form

$$\left(\frac{\partial}{\partial x_k} + \frac{1}{2}\frac{P_k}{P} - \frac{1}{2}\frac{Q_k}{Q} + A_k\right)\left[M_k\left(1 + \frac{Q}{P}\right)\right] = 0, \quad \ldots \tag{3.6}$$

where

$$M_k = \frac{\partial M}{\partial x_k} = \frac{P_k + Q_k}{P+Q} + \frac{\bar{B}^2{}_k}{A_k}. \quad \ldots \tag{3.7}$$

where h is a complex constant of integration, $h = h_0+ih_1$. The general solution of (3.20) may be presented in the well-known form

$$e^{M+q+iS} = h \operatorname{sech}^2(h^{\frac{1}{2}}M+k)/(-ib), \qquad \ldots \quad (3.22)$$

where k is an arbitrary complex constant.

It may now be proved that the real part h_0 of h in (3.22) must be $\frac{1}{4}$.

From (3.21) reverting to the variable $x_k+\epsilon x_l$ we get

$$(M_k+A_k+i\bar{B}_k)^2 = M_k^2[4ib\, e^M\, (\phi_{mn}+i\phi_{mm})+4(h_0+ih_1)].$$

Equating the real parts in the above, we have

$$A^2_k-\bar{B}^2_k+2A_kM_k = -4b\phi_{mm}M^2_k e^M + (4h_0-1)M_k^2, \qquad \ldots \quad (3.23)$$

which, in view of (3.14) and (3.18), gives $h_0 = \frac{1}{4}$.

For the final solution with $\lambda \neq 0$ we have to consider the equations (3.22), (3.12) and (3.10). Thus

$$\phi_{mn}+i\phi_{mm} = e^{-M}\, [h \operatorname{sech}^2(h^{\frac{1}{2}}M+k)/(-ib)], \qquad \ldots \quad (3.24)$$

$$S_{ll}S_{kk} = \frac{\epsilon^2}{C_1^2}\left(\frac{\partial}{\partial x_k}\, e^M\right)^2(\phi^2_{mm} + \phi^2_{mn}), \qquad \ldots \quad (3.25)$$

$$S_{ll} + \epsilon^2 S_{kk} = \frac{1}{C'}\, e^M, \qquad \ldots \quad (3.26)$$

where M is an arbitrary function of $x_k + \epsilon x_l$, k is an arbitrary complex constant, h is a complex constant of the form $\frac{1}{4}+ih_1$ and b, c_1, C' are real arbitrary constants.

The three equations written above give the solution of the field components apparently as functions of $x_k + cx_l$. Consider now the transformation

$$x'_k = x_k + cx_l, \quad x'_l = \phi(x_k, x_l), \qquad (3.27)$$

where

$$\frac{\partial\phi}{\partial x_k} = \frac{-\epsilon S_{kk}}{\epsilon^2 S_{kk} + S_{ll}}, \quad \frac{\partial\phi}{\partial x_l} = \frac{S_{ll}}{\epsilon^2 S_{kk} + S_{ll}},$$

then the above solutions may be presented as involving one variable x'_k and are of the same form as obtained by Bonnor (1951) for the static field.

4. SOLUTION FOR A SPECIAL CASE WITH $\lambda \neq 0$

Returning to the equation (3.6) we notice that it is automatically satisfied if M_k, as defined in (3.7), identically vanishes. Let us consider now the special case with $M_k = 0$. Referring to (3.4) and (3.1) this condition is seen to be equivalent to

$$\frac{\partial}{\partial x_k} \log (S_{ll} + \epsilon^2 S_{kk}) = 0 \qquad \text{... (4.1)}$$

whence $S_{ll} + c^2 S_{kk} = \alpha$ (a constant).

Again, from (3.1) we have

$$\frac{1}{S_{kk}} + \frac{\epsilon^2}{S_{ll}} = \frac{P+Q}{A^2{}_k(\phi^2{}_{mm} + \phi^2{}_{mn})^{\frac{1}{2}}}.$$

Therefore

$$S_{kk} S_{ll} = \frac{\alpha \, . \, A^2{}_k(\phi^2{}_{mm} + \phi^2{}_{mn})^{\frac{1}{2}}}{P+Q} \qquad \text{... (4.2)}$$

Consider now the equations (3.15) and (3.16). We note, first of all, that if L denotes the expression

$$\frac{(S_{kk} S_{ll})^{\frac{1}{2}}}{(\phi^2{}_{mm} + \phi^2{}_{mn})^{\frac{1}{4}}} \qquad \text{... (4.3)}$$

then

$$-\frac{\partial}{\partial x_k} \log L = \frac{P_k + Q_k}{P+Q} - \frac{1}{2}\frac{P_k}{P} - \frac{1}{2}\frac{Q_k}{Q} + A_k.$$

Inserting this and (4.2) in the equations (3.15) and (3.16) we get

$$B_{k;k} - B_k \frac{L_k}{L} = 2\beta \phi_{mn} L^2, \qquad \text{... (4.4)}$$

$$A_{k;k} - A_k \frac{L_k}{L} = -\, 2\beta \phi_{mm} L^2, \qquad \text{... (4.5)}$$

where

$$\beta = \frac{1}{4}\frac{\lambda}{\alpha}.$$

We also find that the equation (2.1) gives

$$A_{k;k} - A_k \frac{L_k}{L} = \frac{1}{2}(A^2{}_k - B^2{}_k) \qquad \text{... (4.6)}$$

Adopting the method of Wyman (1950) to solve the equations (4.4) and (4.5) in conjunction with (4.6) the solution may be presented in the form

$$\phi_{mn} + i\phi_{mm} = h \operatorname{sech}^2 (h^{\frac{1}{2}} x + k)/(-i\beta),$$

$$\frac{\partial x}{\partial x_k} = L, \qquad (4.7)$$

where x is an arbitrary function of $x_k + \epsilon x_l$, k is a complex constant, h is a pure imaginary constant. Further from (4.1) and (4.3) we have

$$S_{ll} + \epsilon^2 S_{kk} = \alpha,$$

$$S_{kk} S_{ll} = \left(\frac{\partial x}{\partial x_k} \right)^2 (\phi^2{}_{mm} + \phi^2{}_{mn}). \qquad \ldots \quad (4.8)$$

The transformation (3.27) applies to this case also and the solutions are found to involve one variable x'_k and thus belong to a static field. Omitting the primes we write the solutions as follows :

$$\phi_{mn} + i\phi_{mm} = h \operatorname{sech}^2(h^{\frac{1}{2}} x + k)/(- i\beta),$$

$$S_{ll} = \alpha, \qquad \ldots \quad (4.9)$$

$$S_{kk} = \frac{1}{\alpha} \left(\frac{dx}{dx_k} \right)^2 (\phi^2{}_{mm} + \phi^2{}_{mn}),$$

where x is an arbitrary function of x_k, h, k, β, α being constants as before. The solutions (4.9) for a static field have been discussed by Bonnor (1951).

5. SOLUTIONS FOR THE CASE $\lambda = 0$

When $\lambda = 0$ the field components are all functions of $x_k + cx_l$ and do not involve x_m (Ghosh, 1956). We shall consider now two cases, (*i*) $P+Q \neq 0$, (ii) $P+Q = 0$. Referring to (3.17) and (3.18) we have for the first case the equations

$$B_{k,k} - \bar{B}_k \frac{M_{k,k}}{M_k} = 0, \qquad \ldots \quad (5.1)$$

$$A_{k,k} - A_k \frac{M_{k,k}}{M_k} = 0. \qquad \ldots \quad (5.2)$$

Using (5.2) in (3.14) we have also

$$A^2{}_k - \bar{B}^2{}_k + 2A_k M_k = 0. \qquad \ldots \quad (5.3)$$

Making use of (3.19) in conjunction with (5.3) we can integrate the equations (5.1,2) and obtain as solution

$$\phi_{mn} + i\phi_{mm} = k\, e^{(\alpha_1 + i\alpha_2)M}, \qquad \ldots \quad (5.4)$$

where M is an arbitrary function of $x_k + \epsilon x_l$, k is an arbitrary complex constant and α_1 and α_2 are two constants connected by the relation

$$\alpha_1{}^2 - \alpha_2{}^2 + 2\alpha_1 = 0. \qquad \ldots \quad (5.5)$$

The components S_{kk}, S_{ll} are given by (3.25) and (3.26). If we now apply the transformation (3.27) the solutions are all transformed into the static form involving one variable x'_k.

In the special case with $M_k = 0$ considered in § 4, we also get a static solution. The equations (4.4, 5) become, when $\lambda = 0$,

$$\bar{B}_{k,k} - \bar{B}_k \frac{L_k}{L} = 0, \qquad \ldots \quad (5.6)$$

$$A_{k,k} - A_k \frac{L_k}{L} = 0 \qquad \ldots \quad (5.7)$$

with

$$A^2_k - \bar{B}_k^2 = 0. \qquad \ldots \quad (5.8)$$

These equations in conjunction with (3.19) will give the solutions for ϕ_{mm} and ϕ_{mn}, while the set of equations (4.8) gives S_{kk}, S_{ll}. If we now apply the transformation (3.27), the first set of solutions is transformed into the static form considered in one of my earlier papers (Ghosh, 1956) and the second set gives the solutions for S_{kk}, S_{ll} as in (4.9).

In the second case, since $P+Q = 0$ we have $S_{ll} + \epsilon^2 S_{kk} = 0$ and consequently $\gamma_{kkk} = \gamma_{llk}$. Substituting these in the original system of equations (2.1—4) and remembering $f(x_m) = 0$ we notice that the last three are automatically satisfied and we are left with the equation (2.1) expressed as

$$- A_{k,k} + A_k \gamma_{lkk} - \tfrac{1}{2}(A^2_k + \bar{B}^2_k) = 0. \qquad \ldots \quad (5.9)$$

This furnishes typical wave solutions. Setting

$$(\phi^2_{mm} + \phi^2_{mn})^{\frac{1}{2}} = u^2, \qquad \phi_{mm}/\phi_{mn} = v,$$

u, v being arbitrary functions of $x_k + \epsilon x_l$ and introducing a function w defined by the equation $\bar{B}^2_k/A_k = 2w_{,k}/w$

that is, $(v_{,k})^2\, u\, w = 4(1 + v^2)^2 u_{,k}\, w_{,k}$

the solutions may be presented as follows :

$$S_{kk} = C' u_{,k}\, w, \qquad S_{ll} = - \epsilon^2 S_{kk}, \qquad \ldots \quad (5.10)$$

$$S_{mm} = S_{nn} = \frac{u^2 v}{\sqrt{1+v^2}}, \quad a_{mn} = \frac{u^2}{\sqrt{1+v^2}}, \qquad \ldots \quad (5.10)$$

The exact wave solutions as obtained above relate to a tensor field having plane symmetry (Taub, 1951). A field with spherical symmetry is found not to admit such wave solutions.

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